Math 618 HW 1 Solution

E. The Hopf map is defined as $f : S^3 \subset \mathbb{C}^2 \to S^2 = \mathbb{CP}^1,\ (z, w) \mapsto [z, w]$. Cover S^2 by two open sets $U = \{[z, w] = [1, \frac{w}{z}] \in \mathbb{CP}^1 | z \neq 0\},\ V = \{[z, w] = [\frac{z}{w}, 1] \in \mathbb{CP}^1 | w \neq 0\}$. Then we define the local trivialization map over U by $\phi_U : f^{-1}(U) \to U \times S^1,\ \phi_U(z, w) = ([1, \frac{w}{z}], \frac{z}{|z|})$. Similarly for $f^{-1}(V)$.

I. The pullback bundle is by definition $p^*\xi = \{(x, y) \in S^3 \times S^3 | p(x) = p(y)\}$. Since $p(x) = p(y)$, there exists a unit complex number $\lambda \in S^1$, so that $y = \lambda x$. Now define the global trivialization map as $p^*\xi \to S^3 \times S^1,\ (x, y) \mapsto (x, \lambda)$.

J. Define a map $h : E_1 \to f_\sharp^* E_2,\ e \mapsto (p_1(e), f(e)) \in B_1 \times E_2$. Since the bundle diagram commutes, the image of h lies in $f_\sharp^* E_2$. Now we construct the inverse map of h. For any element $(x, y) \in f_\sharp^* E_2 \subset B_1 \times E_2$, since $f(x) = p_2(y)$ and f maps the fiber $p_1^{-1}(x)$ in E_1 homeomorphically onto the fiber $p_2^{-1}(f(x))$ in E_2, there exists a unique point $e \in p_1^{-1}(x)$ satisfying $f(e) = y \in p_2^{-1}(f(x))$. This e is the desired preimage of (x, y).

L. Check exactness at $\pi_n(A, x_0)$: an element f in $\pi_n(A, x_0)$ is trivial in $\pi_n(X, x_0)$, iff it admits a filling $D^{n+1} \to X$, which is equivalent to saying that it comes from an element in $\pi_{n+1}(X, A, x_0)$.
Exactness at $\pi_n(X, x_0)$: an element g in $\pi_n(X, x_0)$ is trivial in $\pi_n(X, A, x_0)$ iff g is retractible onto a map $I^n \to A$, which is equivalent to saying it comes from an element in $\pi_n(A, x_0)$.
Exactness at $\pi_n(X, A, x_0)$: if an element h in $\pi_n(X, A, x_0)$ satisfies $\partial h = id\pi_{n-1}(A, x_0)$, then there exists a homotopy in A from ∂h to the point map x_0. Concatenate h with that homotopy, we get a homotopy from h to an element in $\pi_n(X, x_0)$.